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Rotation of Coordinate axes and Differentiation of Integrals with respect to Translation Invariant Bases

AN ABSTRACT
of the dissertation for the academic degree
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Topicality of Research. The study of the influence of a change of variable on analytical properties of a function is one of the important problems in harmonic analysis. The following two main questions were the object of research in this direction: 1) *Is it possible to achieve fulfillment of a given analytical property of a function by means of a change of variable of given type*? 2) *What kind of changes of variable conserves a given analytical property of a function*? In connection with above mentioned problems, the important results were obtained by H. Bohr, A. Beurling and H. Helson, A. Olevskii, J.-P. Kahane and Y. Katznelson, A. Saakyan, U. Jurkat and D. Waterman, A. Baerstein and D. Waterman, in the case, when analytical property of a function is an uniform or absolute convergence of Fourier trigonometric series and a change of variable is a homeomorphism of torus. The possibility of improvement and conservation of function differential properties by means of homeomorphic mapping were studied by E. Brukner and C. Goffman, M. Laczkovich and D. Preiss. Abovementioned results are given in the review work by Olevskii [11] and in the monograph by C. Goffman, T. Nishiura and D. Waterman [4].

The study of the influence of a choice of coordinate axes (i.e. changing a variable, which is a rotation around the origin) on the properties of summable functions of several variables was initiated by A.Zygmund, in particular, he posed a problem concerning the possibility of achieving strong differentiability by means of rotations. The problems on the possibility of achieving and conservation of the properties of strong integral means' convergence, convergence of multiple Fourier series and Fourier integrals in Pringsheim sense and belonging to the classes of functions with bounded variation in various senses in case of rotations, were studied by J. Marstrand, B. Lopez-Melero, A. Stokolos, G. G. Oniani, G. Lepsveridze, G. Karagulyan, M. Dyachenko and O. Dragoshanskii.

The Aim of Dissertation. The proposed dissertation is aimed at: the study of A. Zygmund's problem on possibility of achievement of strong differentiability of an integral by means of a rotation (i.e. choosing coordinate axes) for general classes of bases; investigation of question of conservation of integral differentiation property in case of rotations for translation invariant bases consisting of multi-dimensional intervals; the study of singularities from the standpoint of differentiability of the integral with respect to a given basis, which may have a fixed function for various choices of coordinate axes; the study of differential properties of singular Lebesgue-Stieltjes measures.

Research Novelty.

1) There is given a solution of A. Zygmund's problem for Busemann-Feller and homothecy invariant bases, in particular, it is established, that if a basis of such type is nonstandard (i.e. if it does not differentiate the integral of some summable function), then there exists a function for which the differentiability of integral can not be achieved by means of rotations;
2) It is established that for an arbitrary translation invariant nonstandard basis consisting of multi-dimensional intervals, integral differentiation property is not conserved in case of rotations;
3) It is introduced the definitions of sets of singular rotations. These sets express the singularities, which may have a fixed function for various choices of coordinate axes from the standpoint of differentiability of the integral with respect to a given basis. It is established topological structure of sets of singular rotations;
4) It is given a characterization of not more than countable sets of singular rotations for an arbitrary translation invariant nonstandard basis formed of two-dimensional intervals;

5) It is given a complete characterization of sets of singular rotations for an arbitrary Busemann-Feller, homothecy invariant, nonstandard and symmetric basis formed of two-dimensional intervals.
6) It is established that singular Lebesgue–Stieltjes measures do not have the property of vanishing of means almost everywhere for any translation invariant nonstandard basis.

Approbation of Work. The dissertation results have been presented at V International Conference of the Georgian Mathematical Union (September 8–12, 2014, Batumi, Georgia); VI International Conference of the Georgian Mathematical Union (July 12–16, 2015, Batumi, Georgia); Swedish–Georgian Conference in Analysis & Dynamical Systems (July 15–22, 2015, Tbilisi, Georgia); International Conference "Function Spaces and Function Approximation Theory" dedicated to the 110th anniversary of Academician S. M. Nikolskii (May 25–29, 2015, Moscow, Russia); International Conference "Harmonic Analysis and Integral theory" dedicated to the 80th Jubelee of Professor V.A. Skvortsov (September 23–24, 2015, Moscow, Russia).

Publication. There are published six scientific works, which are listed below the text of this author's abstract.

The size and the structure. The dissertation contains 80 pages. It consists of the introduction, six sections and bibliography. The bibliography contains 32 names.

Content of Dissertation

First let us introduce some definitions and recall some results from the differentiation theory of integrals.

A mapping B defined on $\mathbb{R}^n$ is said to be a *differentiation basis* if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of bounded measurable sets with positive measure and containing x, such that there exists a sequence $R_k \in B(x)$ $(k \in \mathbb{N})$ with $\lim\limits_{k\to\infty} \operatorname{diam} R_k = 0$.

For $f \in L(\mathbb{R}^n)$, the numbers

$$\overline{D}_B\left(\int f, x\right) = \overline{\lim_{\substack{R \in B(x) \\ \operatorname{diam} R \to 0}}} \frac{1}{|R|} \int_R f, \quad \underline{D}_B\left(\int f, x\right) = \underline{\lim_{\substack{R \in B(x) \\ \operatorname{diam} R \to 0}}} \frac{1}{|R|} \int_R f$$

are called *the upper and the lower derivative*, respectively, *of the integral of f at a point x*. If the upper and the lower derivative coincide, then their combined value is called the *derivative of $\int f$ at a point x* and denoted by $D_B(\int f, x)$. We say that the *basis B differentiates $\int f$* (or $\int f$ is differentiable with respect to B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions X we say that B differentiates X.

Denote by $\mathbf{Q} = \mathbf{Q}(\mathbb{R}^n), \mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ and $\mathbf{P} = \mathbf{P}(\mathbb{R}^n)$ the bases of for which:

- $\mathbf{Q}(x)$ $(x \in \mathbb{R}^n)$ consists of all open n-dimensional cubic intervals containing x;
- $\mathbf{I}(x)$ $(x \in \mathbb{R}^n)$ consists of all open n-dimensional intervals containing x;
- $\mathbf{P}(x)$ $(x \in \mathbb{R}^n)$ consists of all open n-dimensional rectangles containing x.

Note that differentiation with respect to $\mathbf{Q}$ and $\mathbf{I}$ are called ordinary and strong differentiation, respectively.

About the bases $\mathbf{Q}$, $\mathbf{I}$ and $\mathbf{P}$ there are known following fundamental results (see e.g. [5]):

The basis of cubes $\mathbf{Q}$ differentiates $L(\mathbb{R}^n)$ (A. Lebesgue, 1910);

The basis of intervals $\mathbf{I}$ differentiates $L(1 + \ln^+ L)^{n-1}(\mathbb{R}^n)$ (B. Jessen, I. Marcinkiewicz and A. Zygmund, 1935);

The basis of intervals $\mathbf{I}$ does not differentiate $L(\mathbb{R}^n)$, moreover, $\mathbf{I}$ does not differentiate any integral class $\varphi(L)(\mathbb{R}^n)$ wider than $L(1 + \ln^+ L)^{n-1}(\mathbb{R}^n)$ (S. Saks, 1935);

The basis of rectangles $\mathbf{P}$ does not differentiate even the class $L^\infty(\mathbb{R}^n) \cap L(\mathbb{R}^n)$ (A. Zygmund, 1927).

For a basis B, we denote by $\overline{B}$ the union of families $B(x)$ $(x \in \mathbb{R}^n)$.

A basis B is called:

translation invariant (briefly, TI-basis) if $B(x) = \{x + R : R \in B(0)\}$ for every $x \in \mathbb{R}^n$;

homothecy invariant (briefly, HI-basis) if for every $x \in \mathbb{R}^n, R \in B(x)$ and a homothethy H with the centre at x we have that $H(R) \in B(x)$;

formed of sets from the class Δ if $\overline{B} \subset \Delta$;

convex if it is formed of the class of all convex sets;

Busemann–Feller basis if it is formed of open sets and the following condition holds: $(x \in \mathbb{R}^n, R \in B(x), y \in R) \Rightarrow R \in B(y)$.

Let us introduce the following notation:

$\mathfrak{B}_{\mathrm{TI}}$ is the class of all translation invariant bases;

$\mathfrak{B}_{\mathrm{HI}}$ is the class of all homothecy invariant bases;

$\mathfrak{B}_{\mathrm{BF}}$ is the class of all Busemann–Feller bases;

$\mathfrak{B}_{\mathrm{NL}}$ is the class of all bases which does not differentiate $L(\mathbb{R}^n)$.

Note that if $B \in \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}}$, then $B \in \mathfrak{B}_{\mathrm{TI}}$ (see e.g. [12, Ch. I, §3]).

A basis B is called *sub-basis of a basis* B'(denoted as $B \subset B'$) if $B(x) \subset B'(x)$ for every $x \in \mathbb{R}^n$. For a basis B by $\mathfrak{B}_B$ we will denote the class of all sub-basis of B.

The *maximal operator M_B and truncated maximal operator M_B^δ $(\delta > 0)$ corresponding to a basis B* are defined as follows

$$M_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \int_R |f|,$$

$$M_B^\delta(f)(x) = \sup_{R \in B(x), \operatorname{diam} R < \delta} \frac{1}{|R|} \int_R |f|,$$

where $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Note that if B is translation invariant or Busemann–Feller basis, then for any f the functions $\overline{D}_B\left(\int f, \cdot\right)$, $\underline{D}_B\left(\int f, \cdot\right)$, $M_B(f)$ and $M_B^\delta(f)$ are measurable (see e.g. [5] or [12]).

In what follows the dimension of the space $\mathbb{R}^n$ is assumed to be greater than 1.

For a basis B by F_B denote the class of all functions $f \in L(\mathbb{R}^n)$ the integrals of which are differentiable with respect to B.

By Γ_n denote the family of all rotations in $\mathbb{R}^n$.

We say that a function f is *reduced in the class F by a transformation of a variable γ* if $f \circ \gamma \in F$.

The work contains six sections.

In the first section of the work it is studied a problem of A. Zygmund concerning a possibility of improvement of function properties by means of choosing of coordinate axes(i.e. by means of a change of variable which is a rotation).

The question on possibility of improvement of a function properties by means of change of variable has quite rich history. Concerning the possibility of improvement Fourier trigonometric series behaviour by means of homeomorphic change of variable there are known important results of H. Bohr, A. Olevskii, J.-P. Kahane and Y. Katznelson, A. Saakyan (see e.g. [11], [4], [18]).

In the integral differentiation theory the study of the above mentioned question was began by the following problem of A. Zygmund (see [5, Ch. IV, §2]): *Can an arbitrary function $f \in L(\mathbb{R}^2)$ be reduced in the class $F_{\mathbf{I}}$ by means of a rotation of coordinate axes?*

J. Marstrand [10] gave the negative answer to the problem by constructing a non-negative function $f \in L(\mathbb{R}^2)$ such that $f \circ \gamma \notin F_{\mathbf{I}}$ for any rotation $\gamma \in \Gamma_2$.

The problem of A. Zygmund in general setting is formulated as follows: *Let B be a translation invariant basis which does not differentiate $L(\mathbb{R}^n)$. Does there exist a function $f \in L(\mathbb{R}^n)$ which can not be reduced in class F_B by means of rotation of coordinate axes?* To formulate the known results in this direction let us introduce some definitions.

For a translation invariant basis B by G. G. Oniani (see [12, Ch. II, §1] or [13]) it was defined the following function

$$\sigma_B(\lambda) = \lim_{t \to \infty} \overline{\lim_{\varepsilon \to 0}} \frac{|\{M_B^{t\varepsilon}(\chi_{V_\varepsilon}) > \lambda\}|}{|V_\varepsilon|} \quad (0 < \lambda < 1),$$

where V_ε is the ball with centre at the origin and with the radius ε. Here and below everywhere χ_E denotes the characteristic function of a set E. We will call σ_B *spherical halo function of B*.

It is easy to check that:

1) If $B \in \mathfrak{B}_{\mathrm{TI}}$ is a convex basis, then due to the following estimation (see [13, Lemma 1]) $M_B(\chi_{V_\varepsilon})(x) \leq C\varepsilon / \operatorname{dist}(x, V_\varepsilon)$ $(x \notin V_{2\varepsilon})$ we have

$$\sigma_B(\lambda) = \overline{\lim_{\varepsilon \to 0}} \frac{|\{M_B(\chi_{V_\varepsilon}) > \lambda\}|}{|V_\varepsilon|};$$

2) If $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{HI}}$, then

$$\sigma_B(\lambda) = |\{M_B(\chi_V) > \lambda\}|,$$

where V is the unit ball.

For a translation invariant basis B by B. Lópes-Melero [9] it was introduced the following weak variant of the spherical halo function

$$\widetilde{\sigma}_B(\lambda) = \overline{\lim_{\varepsilon \to 0}} \frac{|\{M_B^{\varepsilon/\lambda}(\chi_{V_\varepsilon}) > \lambda\}|}{|V_\varepsilon|} \quad (0 < \lambda < 1),$$

Obviously,

$$\widetilde{\sigma}_B(\lambda) \leq \sigma_B(\lambda) \quad (0 < \lambda < 1).$$

We will say that a function $\sigma : (0, 1) \to (0, \infty)$ is *non-regular* if

$$\overline{\lim_{\lambda \to 0}} \lambda \sigma(\lambda) = \infty.$$

For $n \geq 2$ and $2 \leq k \leq n$ by $\mathbf{I}_k^n$ denote the basis for which $\mathbf{I}_k^n(x)$ $(x \in \mathbb{R}^n)$ consists of all open n-dimensional intervals containing x and the lengthes of edges of which take not more than k values. Note that $\mathbf{I}_n^n = \mathbf{I}$.

For a basis B denote by S_B the class of all non-negative functions $f \in L(\mathbb{R}^n)$ such that $\overline{D}_B\left(\int f \circ \gamma, x\right) = \infty$ almost everywhere for every $\gamma \in \Gamma_n$.

By A. Stokolos [19], B. Lópes-Melero [9] and G. G. Oniani [12, Ch. II, §1] (see also [13]), respectively, were established the following results.

Theorem A. *For every $n \geq 2$ and $2 \leq k \leq n$ the class $S_{\mathbf{I}_k^n}$ is non-empty.*

Theorem B. *If a translation invariant basis B has a non-regular weak spherical halo function $\widetilde{\sigma}_B$, then the class S_B is non-empty.*

Theorem C. *If a translation invariant basis B has a non-regular spherical halo function σ_B, then the class S_B is non-empty.*

Note that from the estimations: $\widetilde{\sigma}_B(\lambda) \le \sigma_B(\lambda)$ and $\widetilde{\sigma}_{\mathbf{I}_k^n}(\lambda) \ge c\frac{1}{\lambda}\ln^{k-1}\frac{1}{\lambda}$ it follows the implications: Theorem C $\Rightarrow$ Theorem B $\Rightarrow$ Theorem A.

The following theorem gives the answer to Zygmund's generalized problem for the class of bases $\mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$.

Theorem 1.1. *If $B \in \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then the class S_B is non-empty.*

Theorem 1.1 we prove using Theorem C on the basis of the following Lemma.

Lemma 1.4. *If $B \in \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then B has a non-regular spherical halo function.*

In the second section the question on the invariance of classes of functions with differentiable integrals with respect to the class of transformations of variable consisting of all rotations is studied.

A class of functions with good analytical properties may be very sensitive with respect to changes of variable. Let us recall the result of such type belonging to A. Beurling and H. Helson [1]: *Let $\mathbb{T}$ be the unit circumference on the complex plane and $A(\mathbb{T})$ be the class of all continuous on $\mathbb{T}$ functions having absolutely convergent Fourier trigonometric series. For a homeomorphism $\gamma : \mathbb{T} \to \mathbb{T}$ we have that $f \in A(\mathbb{T}) \Rightarrow f \circ \gamma \in A(\mathbb{T})$ if and only if γ is of the type $\gamma(e^{it}) = e^{i(kt+a)}$, where $k \in \{-1, 1\}$ and $a \in [-\pi, \pi]$.*

A class of functions F is called *invariant with respect to a class of transformations of a variable* Γ if $(f \in F, \gamma \in \Gamma) \Rightarrow f \circ \gamma \in F$.

Thus the only homeomorphisms with respect to which the class $A(\mathbb{T})$ is invariant are rotations, conjugation and their compositions. In particular, there exists a dipheomorpism $\gamma : \mathbb{T} \to \mathbb{T}$ with respect to which $A(\mathbb{T})$ is not invariant.

The dependence of the properties of functions of several variables on a choice of coordinate axes (i.e. on a rotation of the standard orthogonal coordinate system) were studied by different authors.

From the results of G. Lepsveridze [8], G. G. Oniani [14] and A. Stokolos [20] it follows that the class $F_{\mathbf{I}}$ is not invariant with respect to linear changes of a variable, in particular with respect to rotations. An analogous result was established by O. Dragoshanskii [2] for the class of continuous functions of two variables, having an a.e. converging Fourier series (Fourier integral) in Pringsheim sense.

G. Karagulyan [6] gave, in the two-dimensional case, a complete characterization of singularities from the standpoint of differentiability with respect to a basis $\mathbf{I}$ which may have the integral of a fixed function for various choices of a coordinate system. The multi-dimensional aspect of this question was studied by G. G. Oniani [15].

M. Dyachenko [3] considered a problem of invariance with respect to Γ_2 of two-dimensional classes of functions with bounded variation in various senses.

The result on the non-invariance of the class $F_{\mathbf{I}}$ with respect to rotations can be extended to bases of quite general type. In particular, the following theorem is true.

Theorem 2.1. *If $B \in \mathfrak{B}_{\mathbf{I}} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then the class F_B is not invariant with respect to rotations, moreover, there exists a non-negative function $f \in F_{\mathbf{I}}$ such that $f \circ \gamma \notin F_B$ for some $\gamma \in \Gamma_n$.*

Note that if B differentiates $L(\mathbb{R}^n)$, then the question on the invariance of the class F_B with respect to rotations is trivial, in particular, taking into account that a rotation is measure preserving mapping we conclude the invariance of F_B with respect to rotations.

In the third section there are introduced definitions of sets of singular rotations and there are established some results concerning their structure.

Let B be a basis in $\mathbb{R}^n$ and $\gamma \in \Gamma_n$. The *γ-rotated basis B* is defined as follows

$$B(\gamma)(x) = \{x + \gamma(R - x) : R \in B(x)\} \quad (x \in \mathbb{R}^n).$$

Suppose B is translation invariant. Then it is easy to verify that the differentiation of the integral of a "rotated" function $f\circ\gamma$ with respect to B at a point x is equivalent to the differentiation of the integral of f with respect to the "rotated" basis $B(\gamma^{-1})$ at a point $\gamma^{-1}(x)$. Consequently, we can reduce the study of the behavior of functions $f\circ\gamma$ $(\gamma\in\Gamma_n)$ with respect to the basis B to the study of the behavior of f with respect to rotated bases $B(\gamma)$ $(\gamma\in\Gamma_n)$.

Let B be a basis from the class $\mathfrak{B}_{\mathbf{I}}\cap\mathfrak{B}_{\mathrm{TI}}\cap\mathfrak{B}_{\mathrm{NL}}$. By virtue of Theorem 2.1 there exists a function having a non-homogeneous behaviour with respect to rotated bases $B(\gamma)$ $(\gamma\in\Gamma_n)$, more exactly, $\int f$ is not differentiable with respect to $B(\gamma)$ for some rotations and $\int f$ is differentiable with respect $B(\gamma)$ for some γ rotations. Thus for f some rotations γ are "singular" (non-differentiability with respect to $B(\gamma)$) and some rotations γ are "regular" (differentiability with respect to $B(\gamma)$). In this connection naturally arises problem: *what kind of may be sets of singular and of regular rotations for a fixed function*? Note that by duality argument we can restrict ourselves by studying sets of singular rotations.

The posed problem for the case of strong differentiability process (i.e., for the case $B=\mathbf{I}$) was studied in works of G. Karagulyan [6], G. G. Oniani [12, 14, 15], G. Lepsveridze [8] and A. Stokolos [20].

In connection to the posed problem let us introduce rigor definition of a set of singular rotations: Suppose B is a basis in $\mathbb{R}^2$ and $E\subset\Gamma_2$. Let us call E a *W_B-set* if there exists a function $f\in L(\mathbb{R}^2)$ with the following two properties:

$$f\notin F_{B(\gamma)}\ \text{ for every }\ \gamma\in E;$$
$$f\in F_{B(\gamma)}\ \text{ for every }\ \gamma\notin E.$$

Let us introduce also the definition of a set of "strongly" singular rotations: Suppose B is a basis in $\mathbb{R}^2$ and $E\subset\Gamma_2$. Let us call E an *R_B-set* if there exists a function $f\in L(\mathbb{R}^2)$ with the following two properties:

$$\overline{D}_{B(\gamma)}\left(\textstyle\int f,x\right)=\infty \text{ a.e. for every } \gamma\in E;$$
$$f\in F_{B(\gamma)} \text{ for every } \gamma\notin E.$$

It is clear that each R_B-set is W_B-set.

When $B=\mathbf{I}$ we will use terms W-set and R-set. The definitions of an R-set and of a W-set were introduced in [14] and [6], respectively.

Now the problem can be formulated as follows: *For a given basis B what kind of sets are W_B-sets (R_B-sets)*?

The following theorems give necessary conditions of topological character for sets of singular rotations.

Theorem 3.1. *For arbitrary translation invariant basis B in $\mathbb{R}^2$ each W_B-set has $G_{\delta\sigma}$ type.*

Theorem 3.2. *For arbitrary translation invariant basis B in $\mathbb{R}^2$ each R_B-set has G_δ type.*

For the case of W and R sets Theorems 3.1 and 3.2 were proved by G. Karagulyan [6] and G. G. Oniani [14], respectively.

Let us introduce the following generalizations of notions of a W_B-set and of an R_B-set:

Let B and H are bases in $\mathbb{R}^n$ with $B\subset H$ and $E\subset\Gamma_n$. Let us call E a *$W_{B,H}$-set* (*$W^+_{B,H}$-set*), if there exists a function $f\in L(\mathbb{R}^n)$ $(f\in L(\mathbb{R}^n), f\geq 0)$ with the following two properties:

$$f\notin F_{B(\gamma)}\ \text{ for every }\ \gamma\in E;$$
$$f\in F_{H(\gamma)}\ \text{ for every }\ \gamma\notin E.$$

Let B and H are bases in $\mathbb{R}^n$ with $B\subset H$ and $E\subset\Gamma_n$. Let us call E an *$R_{B,H}$-set* (*$R^+_{B,H}$-set*), if there exists a function $f\in L(\mathbb{R}^n)$ $(f\in L(\mathbb{R}^n), f\geq 0)$ with the following two properties:

$$\overline{D}_{B(\gamma)}\left(\textstyle\int f,x\right)=\infty \text{ a.e. for every } \gamma\in E;$$
$$f\in F_{H(\gamma)} \text{ for every } \gamma\notin E.$$

If $B=H$, then instead of $W^+_{B,B}$-set ($R^+_{B,B}$-set) we will use term W^+_B-set (R^+_B-set).

Remark 3.1. It is clear that:

1) If $B = H$, then the notions of $W_{B,B}$-set ($R_{B,B}$-set) and of W_B-set (R_B-set) coincide;

2) each $W^+_{B,H}(R^+_{B,H})$-set is $W_{B,H}(R_{B,H})$-set;

3) each $W_{B,H}(W^+_{B,H}, R_{B,H}, R^+_{B,H})$-set is $W_B(W^+_B, R_B, R^+_B)$-set.

Remark 3.2. Taking into account conclusions of Remark 3.1 from Theorems 3.1 and 3.2 we have: If B and H are translation invariant bases with $B \subset H$, then:

1) each $W_{B,H}$-set is of $G_{\delta,\sigma}$ type;

2) each $R_{B,H}$-set is of G_δ type.

Consequently, taking into account Remark 3.1 again we have also that: each $W^+_{B,H}$-set and each W^+_B-set is of $G_{\delta,\sigma}$ type; each $R^+_{B,H}$-set and each R^+_B-set is of G_δ type.

Theorem 3.3. *For arbitrary bases B and H with $B \subset H$ not more than countable union of $R_{B,H}$-sets ($R^+_{B,H}$-sets) is $W_{B,H}$-set ($W^+_{B,H}$-set).*

For non-empty sets $E_1 \subset \Gamma_2$ and $E_2 \subset \Gamma_2$ denote $E_1E_2 = \{\gamma_1 \circ \gamma_2 : \gamma_1 \in E_1, \gamma_2 \in E\}$. A set $E \subset \Gamma_2$ let us call symmetric if $E = \Pi E$.

A basis B in $\mathbb{R}^2$ let us call *symmetric*, if $B(\gamma) = B$ for every $\gamma \in \Pi$. Note that the basis $\mathbf{I}(\mathbb{R}^2)$ is symmetric.

Remark 3.3. Let bases B and H with $B \subset H$ are given. It is easy to see that if B is symmetric, then each $W_{B,H}(W^+_{B,H}, R_{B,H}, R^+_{B,H})$-set is symmetric.

In the forth section some classes of sets of singular rotations are found. From obtained results it follows a characterization of not more than countable sets of singular rotations for symmetric bases from the class $\mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$.

Theorem 4.1. *Let $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then for every not more than countable set $E \subset \Gamma_2$ and for every sequence of its neighbourhoods (V_k) there is a non-negative function $f \in L(\mathbb{R}^2)$ such that:*

1) For every $\gamma \in E$, $\overline{D}_{B(\gamma)}(\int f, x) = \infty$ almost everywhere;

2) For every $k \in \mathbb{N}$, $f \in F_{\mathbf{I}(\Gamma_2 \setminus \Pi V_k)}$. Consequently, for every $\gamma \notin \bigcap_{k=1}^{\infty} \Pi V_k$ we have that $f \in F_{\mathbf{I}(\gamma)}$;

3) If for $\gamma \in \Gamma_2$ the condition $\overline{D}_{B(\gamma)}(\int f, x) = \infty$ is valid for points from some set of positive measure, then the same condition is valid almost everywhere.

Corollary 4.1. *Let $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then:*

1) every not more than countable symmetric set $E \subset \Gamma_2$ is a $W^+_{B,\mathbf{I}}$-set;

2) every not more than countable symmetric set of G_δ type is a $R^+_{B,\mathbf{I}}$-set.

Taking into account Theorems 3.1 and 3.2 from Corollary 4.1 we derive the following result.

Corollary 4.2. *Let B is a symmetric basis from the class $\mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then:*

1) not more than countable set $E \subset \Gamma_2$ is a $W_{B,\mathbf{I}}$-set ($W^+_{B,\mathbf{I}}$-set) if and only if E is symmetric;

2) not more than countable set $E \subset \Gamma_2$ is a $R_{B,\mathbf{I}}$-set ($R^+_{B,\mathbf{I}}$-set) if and only if E is symmetric and of G_δ type.

From Theorem 4.1 we also derive the following result.

Corollary 4.3. *Let $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then there is a non-negative function $f \in L(\mathbb{R}^2)$ for which the set*

$$\{\gamma \in \Gamma_2 : \overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty \ \ a.e.\}$$

is of the second category and consequently, of the continuum cardinality.

Corollary 4.4. *Let $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then there is a $W^+_{B,\mathbf{I}}$-set of the second category and consequently, of the continuum cardinality.*

Let us say that a basis B has *weak Besikovitch property* if for every non-negative function $f \in L(\mathbb{R}^n)$ the set

$$\{x : f(x) < \overline{D}_B(\textstyle\int f, x) < \infty\}$$

is of zero measure. Note that by virtue of the result M. de Guzmán and M. Menárguez (see [5, Ch. IV, §3]) every basis $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}}$ has weak Besikovitch property. Here we note that in [17] it is found more general class of bases $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)}$ having weak Besikovitch property.

The next assertion follows from Corollary 4.3.

Corollary 4.5. *Let $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. If, additionally, it is known that B has weak Besikovitch property, then there is a $R^+_{B,\mathbf{I}}$-set of the second category and consequently, of the continuum cardinality.*

Since for any basis $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)}$ each $W_{B,\mathbf{I}}(W^+_{B,\mathbf{I}}, R_{B,\mathbf{I}}, R^+_{B,\mathbf{I}})$-set is W_B (W^+_B, R_B, R^+_B)-set, Corollaries 4.1–4.5 imply corresponding results for W_B (W^+_B, R_B, R^+_B)-sets, in particular, Corollary 4.2 implies the following result.

Corollary 4.6. *Let B is a symmetric basis from the class $\mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then:*

1) not more than countable set $E \subset \Gamma_2$ is a W_B-set (W^+_B-set) if and only if E is symmetric;

2) not more than countable set $E \subset \Gamma_2$ is a R_B-set (R^+_B-set) if and only if E is symmetric and of G_δ type.

Theorem 4.1 and its corollaries given above for the case $B = \mathbf{I}$ were proved in [14].

In the fifth section we give a complete characterization of W_B-sets and R_B-sets for a quite wide class of bases

G. Karagulyan [6] gave complete characterization of W-sets and R-sets, namely, in [6] it was proved that:

1) *a set $E \subset \Gamma_2$ is W-set if and only if E is symmetric and of $G_{\delta\sigma}$ type.*

2) *a set $E \subset \Gamma_2$ is R-set if and only if E is symmetric and of G_δ type.*

Developing the scheme of proof suggested in [6], below we establish a characterization of W_B-sets and R_B-sets for a quite wide class of bases.

It is true the following theorem.

Theorem 5.1. *If a basis $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$ has non-regular spherical halo function, in particular, if $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$ (see Lemma 1.4), then:*

1) every symmetric set $E \subset \Gamma_2$ of $G_{\delta\sigma}$ type is $W_{B,\mathbf{I}}$-set;

2) every symmetric set $E \subset \Gamma_2$ of G_δ type is $R_{B,\mathbf{I}}$-set.

The first statement of Theorem 5.1 we derive from the second one on the basis of Theorem 3.3.

Taking into account Theorems 3.1 and 3.2 from Theorem 5.1 we obtain the following result.

Corollary 5.1. *If a symmetric basis $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$ has non-regular spherical halo function, in particular, if B is symmetric and $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then:*

1) a set $E \subset \Gamma_2$ is $W_{B,\mathbf{I}}$-set if and only if E is symmetric and of $G_{\delta\sigma}$ type;

2) a set $E \subset \Gamma_2$ is $R_{B,\mathbf{I}}$-set if and only if E is symmetric and of G_δ type.

Corollary 5.2. *If a symmetric basis $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$ has non-regular spherical halo function, in particular, if B is symmetric and $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then:*

1) a set $E \subset \Gamma_2$ is W_B-set if and only if E is symmetric and of $G_{\delta\sigma}$ type;

2) a set $E \subset \Gamma_2$ is R_B-set if and only if E is symmetric and of G_δ type.

The function constructed in proof of Theorem 5.1 take values both of positive and negative sign. Therefore the method of proof of Theorem 5.1 does not allow us to characterize W^+_B-sets and R^+_B-sets. Note that the problem of characterizing of W^+_B-sets and R^+_B-sets remains open even for the case $B = \mathbf{I}$.

In the sixth section we construct singular Lebesgue-Stieltjes measures having non-vanishing means almost everywhere.

For a Lebesgue–Stieltjes measure μ and a basis B, the numbers

$$\overline{D}_B(\mu, x) = \overline{\lim_{R\in B(x),\, \mathrm{diam}\, R\to 0}} \frac{\mu(R)}{|R|},$$

$$\underline{D}_B(\mu, x) = \underline{\lim_{R\in B(x),\, \mathrm{diam}\, R\to 0}} \frac{\mu(R)}{|R|}$$

are called the upper and the lower derivative with respect to B, respectively, of μ at a point x. If the upper and the lower derivative coincide, then their common value is called a derivative with respect to B of μ at a point x and denoted by $D_B(\mu, x)$.

A basis B is said to differentiate a Lebesgue–Stieltjes measure μ if $D_B(\mu, x)$ exists for almost all $x \in \mathbb{R}^n$;

A Lebesgue–Stieltjes measure μ is called:

- singular if there is a Borel set E such that: $|E| = 0$ and $\mu(A) = \mu(A \cap E)$ for every Borel set A;
- discrete if it has the form: $\mu = \sum_{k\in\mathbb{N}} m_k \delta_{a_k}$, where $m_k \geq 0$ and δ_{a_k} is the Dirac measure supported at a point a_k.

It is obvious that each discrete Lebesgue–Stieltjes measure is singular.

It is well known that (see e.g. [21, Ch. V, §7]) if μ is a singular Lebesgue–Stieltjes measure μ, then

$$D_{\mathbf{Q}}(\mu, x) = 0 \ \text{ almost everywhere.}$$

Singular Lebesgue–Stieltjes measures lose the "vanishing" property if $\mathbf{Q}$ is replaced by an arbitrary translation invariant basis which does not differentiate $L(\mathbb{R}^n)$. Moreover, it is true the following result.

Theorem 6.1. *For every translation invariant basis B which does not differentiate $L(\mathbb{R}^n)$ there exists a discrete finite Lebesgue–Stieltjes measure μ such that*

$$\overline{D}_B(\mu, x) = \infty \ \text{ almost everywhere.}$$

For the case $B = \mathbf{I}$, Theorem 6.1 follows from a result of G. Karagulyan [7] about random measures in $\mathbb{R}^n$.

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